

Supplementary material for Inference of partial correlations of a multivariate Gaussian time series

BY A. S. DiLERNIA

Department of Statistics, Grand Valley State University, Allendale, Michigan 49401, U.S.A.
dilerand@gvsu.edu

5

M. FIECAS AND L. ZHANG

Division of Biostatistics, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.
mfiecas@umn.edu zhan4800@umn.edu

SUMMARY

This supplementary document provides proofs of the results of the main manuscript and explicit forms of relevant gradient vectors and Hessian matrices.

10

1. PROOFS OF THEOREMS 1, 2, AND 3

We provide proofs of Theorem 1 through Theorem 3 of the main manuscript below.

Proof of Theorem 1. We observe that since $x(t)$ is an ergodic, stationary, and stochastic process, $e(t) = \{e_{i \cdot (ij)}(t)\}_{i \neq j}$ is also an ergodic, stationary, and stochastic process (Samorodnitsky, 2016). Moreover, since each component of $x(t)$ has a square integrable spectral density function and a linear combination of square integrable functions is also square integrable, it follows that the spectral density functions for the ordinary least squares residuals are also square integrable, so Condition 1 is met for $e(t)$. Similarly, a linear function of a linear process with finite fourth-order moments is also a linear process with finite fourth-order moments, so Conditions 2 and 3 are satisfied for $e(t)$. Hence, the empirical partial correlations $\{r_{ij \cdot (ij)}\}_{i \neq j}$, which are equivalent to the marginal correlations between the ordinary least squares residuals, are jointly asymptotically normal with mean the population partial correlation (Roy, 1989; Roy and Cl  roux, 1993). $\square$

15

20

Proof of Theorem 2. To derive the asymptotic covariance structure for the p choose 2 unique partial correlations, we use a Taylor series approximation of the empirical partial correlations and properties of quadratic forms of multivariate normal random vectors. First, we observe that one way to express the empirical partial correlation between x_i and x_j is

25

$$r_{ij \cdot (ij)} = f(e_{ij}) = e_{i \cdot (ij)}^T e_{j \cdot (ij)} (e_{i \cdot (ij)}^T e_{i \cdot (ij)} e_{j \cdot (ij)}^T e_{j \cdot (ij)})^{-1/2}, \quad (1)$$

where $e_{ij} = [e_{i \cdot (ij)}^T, e_{j \cdot (ij)}^T]^T$ and $r_{ij \cdot (ij)}$ is equivalent to the sample marginal correlation between $e_{i \cdot (ij)}$ and $e_{j \cdot (ij)}$. We approximate $f(e_{ij})$ in Equation (1) using a second-order Taylor series expansion around $\varepsilon_{ij} = [\varepsilon_{i \cdot (ij)}^T, \varepsilon_{j \cdot (ij)}^T]^T$, the theoretical residuals such that $\rho_{ij \cdot (ij)} = f(\varepsilon_{ij})$, as

$$f(e_{ij}) \approx f(\varepsilon_{ij}) + (e_{ij} - \varepsilon_{ij})^T \nabla f(\varepsilon_{ij}) + 1/2 (e_{ij} - \varepsilon_{ij})^T H\{f(\varepsilon_{ij})\} (e_{ij} - \varepsilon_{ij}),$$

where $\nabla f(\varepsilon_{ij}) = E\{\nabla f(e_{ij})\} \in \mathbb{R}^{2N}$ is the expected value of the gradient of $f(e_{ij})$, and $H\{f(\varepsilon_{ij})\} = E\{H(e_{ij})\} \in \mathbb{R}^{2N \times 2N}$ is the expected value of the Hessian matrix of $f(e_{ij})$. Details regarding the form of the gradient vector and Hessian matrix are provided in Section 2. Since $\nabla f(\varepsilon_{ij}) = 0$, the first term of the Taylor series is 0. Therefore, a second-order approximation of the variance of $f(e_{ij}) = r_{ij \cdot (ij)}$ is $\tilde{\gamma}_{ij} = 1/4 \text{var}[(e_{ij} - \varepsilon_{ij})^T H\{f(\varepsilon_{ij})\}(e_{ij} - \varepsilon_{ij})]$.

Observe that $x_i = X_{(ij)}\hat{\beta}_{i \cdot (ij)} + e_{i \cdot (ij)}$, where $\hat{\beta}_{i \cdot (ij)} = (X_{(ij)}^T X_{(ij)})^{-1} X_{(ij)}^T x_i$. Given Conditions 1 through 6, $\hat{\beta}_{i \cdot (ij)}$ is a consistent estimator for $\beta_{i \cdot (ij)}$ where $x_i = X_{(ij)}\beta_{i \cdot (ij)} + \varepsilon_{i \cdot (ij)}$ (Drygas, 1976). Since $e_{i \cdot (ij)} = x_i - X_{(ij)}\hat{\beta}_{i \cdot (ij)}$ and $x(t)$ is Gaussian, by the continuous mapping theorem $e_{i \cdot (ij)}$ converges in distribution to a Gaussian distribution for all $i \neq j$, and thus $e_{i \cdot (ij)} - \varepsilon_{i \cdot (ij)}$ converges in distribution to a mean 0 Gaussian distribution. By the variance of quadratic forms of mean 0 Gaussian random vectors, it follows that $\tilde{\gamma}_{ij} = 1/2 \text{tr}[H\{f(\varepsilon_{ij})\}\Sigma_{ij}H\{f(\varepsilon_{ij})\}\Sigma_{ij}]$, where $\text{tr}(\cdot)$ denotes the trace function and $\Sigma_{ij} = \text{cov}(e_{ij}) \in \mathbb{R}^{2N \times 2N}$ is

$$\Sigma_{ij} = \begin{bmatrix} \text{cov}(e_{i \cdot (ij)}) & \text{cov}(e_{i \cdot (ij)}, e_{j \cdot (ij)}) \\ \text{cov}(e_{j \cdot (ij)}, e_{i \cdot (ij)}) & \text{cov}(e_{j \cdot (ij)}) \end{bmatrix}.$$

Extending this result to obtain the asymptotic covariance matrix for any pair of partial correlations, we let $e_{ijk m} = [e_{i \cdot (ij)}^T, e_{j \cdot (ij)}^T, e_{k \cdot (km)}^T, e_{m \cdot (km)}^T]^T \in \mathbb{R}^{4N}$, and we consider the function

$$g(e_{ijk m}) = \begin{bmatrix} f(e_{ij}) \\ f(e_{km}) \end{bmatrix} = \begin{bmatrix} r_{ij \cdot (ij)} \\ r_{km \cdot (km)} \end{bmatrix},$$

such that $g : \mathbb{R}^{4N} \rightarrow \mathbb{R}^2$. Through a similar process as in the single partial correlation case, we observe that the asymptotic covariance matrix of $g(e_{ijk m})$ is

$$\text{cov}([r_{ij \cdot (ij)} \ r_{km \cdot (km)}]^T) \approx 1/4 \begin{bmatrix} \text{var}(e_{ij}^T H\{f(\varepsilon_{ij})\}e_{ij}) & \text{cov}(e_{ij}^T H\{f(\varepsilon_{ij})\}e_{ij}, e_{km}^T H\{f(\varepsilon_{km})\}e_{km}) \\ \text{cov}(e_{km}^T H\{f(\varepsilon_{km})\}e_{km}, e_{ij}^T H\{f(\varepsilon_{ij})\}e_{ij}) & \text{var}(e_{km}^T H\{f(\varepsilon_{km})\}e_{km}) \end{bmatrix}.$$

We have already shown the diagonal entries in the single partial correlation case. For the off-diagonal terms, we observe that

$$\text{cov}[e_{ij}^T H\{f(\varepsilon_{ij})\}e_{ij}, e_{km}^T H\{f(\varepsilon_{km})\}e_{km}] = 1/2 (\text{var}(e_{ijk m}^T H_{ijk m} e_{ijk m}) - \text{var}[e_{ij}^T H\{f(e_{ij})\}e_{ij}] - \text{var}[e_{km}^T H\{f(e_{km})\}e_{km}]),$$

where $H_{ijk m} = \begin{bmatrix} H\{f(e_{ij})\} & 0 \\ 0 & H\{f(e_{km})\} \end{bmatrix} \in \mathbb{R}^{4N \times 4N}$. This is since

$$\begin{aligned} \text{var}(e_{ijk m}^T H_{ijk m} e_{ijk m}) &= \text{var}[e_{ij}^T H\{f(e_{ij})\}e_{ij} + e_{km}^T H\{f(e_{km})\}e_{km}], \\ &= \text{var}[e_{ij}^T H\{f(e_{ij})\}e_{ij}] + \text{var}[e_{km}^T H\{f(e_{km})\}e_{km}] \\ &\quad + 2\text{cov}[e_{ij}^T H\{f(e_{ij})\}e_{ij}, e_{km}^T H\{f(e_{km})\}e_{km}]. \end{aligned}$$

Again, by the variance of a quadratic form of a multivariate normal random vector,

$$\text{var}(e_{ijk m}^T H_{ijk m} e_{ijk m}) = 2\text{tr}(H_{ijk m} \Sigma_{ijk m} H_{ijk m} \Sigma_{ijk m}),$$

where $\Sigma_{ijk m} = \text{cov}(e_{ijk m}) = \begin{bmatrix} \text{cov}(e_{ij}) & \Sigma_{ijk m 12} \\ \Sigma_{ijk m 12}^T & \text{cov}(e_{km}) \end{bmatrix} \in \mathbb{R}^{4N \times 4N}$, and

$$\Sigma_{ijk m 12} = \begin{bmatrix} \text{cov}(e_{i \cdot (ij)}, e_{k \cdot (km)}) & \text{cov}(e_{i \cdot (ij)}, e_{m \cdot (km)}) \\ \text{cov}(e_{j \cdot (ij)}, e_{k \cdot (km)}) & \text{cov}(e_{j \cdot (ij)}, e_{m \cdot (km)}) \end{bmatrix} \in \mathbb{R}^{2N \times 2N}.$$

Thus, the asymptotic covariance between $r_{ij \cdot (ij)}$ and $r_{km \cdot (km)}$ is

$$\tilde{\gamma}_{ijk m} = 1/2 \text{tr} \left[H\{f(\varepsilon_{ij})\} \Sigma_{ijk m 12} H\{f(\varepsilon_{km})\} \Sigma_{ijk m 12}^T \right]. \quad \square$$

Proof of Theorem 3. Let $x_i = X_{(ij)} \beta_{i \cdot (ij)} + \varepsilon_{i \cdot (ij)}$ and $\hat{\beta}_{i \cdot (ij)} = \left(X_{(ij)}^T X_{(ij)} \right)^{-1} X_{(ij)}^T x_i$ for $i \neq j$. To show that $\hat{\gamma}_{ijk m} = 1/2 \text{tr} \left[H\{f(e_{ij})\} \hat{\Sigma}_{ijk m 12} H\{f(e_{km})\} \hat{\Sigma}_{ijk m 12}^T \right]$ is a consistent estimator for $\tilde{\gamma}_{ijk m}$ under mild conditions, where $\hat{\Sigma}_{ijk m 12}$ is a tapered covariance estimator based on the empirical ordinary least squares residuals, we will show that $H\{f(e_{ij})\}$ is a consistent estimator for $H\{f(\varepsilon_{ij})\}$, $H\{f(e_{km})\}$ is a consistent estimator for $H\{f(\varepsilon_{km})\}$, and that $\hat{\Sigma}_{ijk m 12}$ is a consistent estimator for $\Sigma_{ijk m 12}$. We first observe that given Conditions 4 through 6, $\hat{\beta}_{i \cdot (ij)}$ converges in probability to $\beta_{i \cdot (ij)}$ (Drygas, 1976). Therefore, by the continuous mapping theorem $e_{i \cdot (ij)}$ converges in probability to $\varepsilon_{i \cdot (ij)}$, and assuming analogous conditions are true for $\varepsilon_{j \cdot (ij)}$, it follows that $e_{j \cdot (ij)}$ converges in probability to $\varepsilon_{j \cdot (ij)}$. Thus, also by the continuous mapping theorem $H\{f(e_{ij})\}$ is a consistent estimator for $H\{f(\varepsilon_{ij})\}$. Through a similar process and set of conditions for $\varepsilon_{k \cdot (km)}$ and $\varepsilon_{m \cdot (km)}$, it follows that $H\{f(e_{km})\}$ is a consistent estimator for $H\{f(\varepsilon_{km})\}$. 45

Also observe that given the conditions of Theorem 1, $x(t)$ is a stationary causal linear process. Since a linear combination of stationary causal linear processes is also a stationary causal linear process, the empirical ordinary least squares residuals satisfy the short-range dependence measure of Wu and Pourahmadi (2009), and thus by the result of McMurphy and Politis (2010), $\hat{\Sigma}_{ijk m 12}$ is a consistent estimator for $\Sigma_{ijk m 12}$. Hence, given Conditions 1 through 6, by the continuous mapping theorem $\hat{\gamma}_{ijk m}$ is a consistent estimator for $\tilde{\gamma}_{ijk m}$. 55

2. DERIVATION OF GRADIENT AND HESSIAN MATRIX

We provide an explicit form of the gradient vector and Hessian matrix in Section 1. For the gradient of $f(\varepsilon_{ij})$, we observe that

$$\begin{aligned} \frac{\partial f}{\partial \varepsilon_{i \cdot (ij)k}} &= \frac{\partial}{\partial \varepsilon_{i \cdot (ij)k}} \{ \varepsilon_{i \cdot (ij)}^T \varepsilon_{j \cdot (ij)} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-1/2} \}, \\ &= \frac{\partial}{\partial \varepsilon_{i \cdot (ij)k}} \left\{ \sum_{t=1}^N \varepsilon_{i \cdot (ij)t} \varepsilon_{j \cdot (ij)t} \left(\sum_{t=1}^N \varepsilon_{i \cdot (ij)t}^2 \sum_{t=1}^N \varepsilon_{j \cdot (ij)t}^2 \right)^{-1/2} \right\}, \\ &= \varepsilon_{j \cdot (ij)k} \left(\sum_{t=1}^N \varepsilon_{i \cdot (ij)t}^2 \sum_{t=1}^N \varepsilon_{j \cdot (ij)t}^2 \right)^{-1/2} \\ &\quad - \varepsilon_{i \cdot (ij)k} \left(\sum_{t=1}^N \varepsilon_{i \cdot (ij)t}^2 \sum_{t=1}^N \varepsilon_{j \cdot (ij)t}^2 \right)^{-3/2} \left(\sum_{t=1}^N \varepsilon_{j \cdot (ij)t}^2 \sum_{t=1}^N \varepsilon_{i \cdot (ij)t} \varepsilon_{j \cdot (ij)t} \right), \end{aligned} \quad \text{65}$$

and similarly

$$\begin{aligned}
\frac{\partial f}{\partial \varepsilon_{k \cdot (ij)k}} &= \varepsilon_{i \cdot (ij)k} \left(\sum_{t=1}^N \varepsilon_{i \cdot (ij)t}^2 \sum_{t=1}^N \varepsilon_{j \cdot (ij)t}^2 \right)^{-1/2} \\
&\quad - \varepsilon_{j \cdot (ij)k} \left(\sum_{t=1}^N \varepsilon_{i \cdot (ij)t}^2 \sum_{t=1}^N \varepsilon_{j \cdot (ij)t}^2 \right)^{-3/2} \left(\sum_{t=1}^N \varepsilon_{i \cdot (ij)t}^2 \right) \sum_{t=1}^N \varepsilon_{i \cdot (ij)t} \varepsilon_{j \cdot (ij)t}.
\end{aligned}$$

Hence, it follows that the gradient of $f(\varepsilon_{ij})$ is

$$\nabla f(\varepsilon_{ij}) = \begin{bmatrix} \frac{\partial f}{\partial \varepsilon_{i \cdot (ij)}} \\ \frac{\partial f}{\partial \varepsilon_{j \cdot (ij)}} \end{bmatrix} = \begin{bmatrix} \frac{\varepsilon_{j \cdot (ij)}}{(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{1/2}} - \frac{\varepsilon_{i \cdot (ij)} (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) (\varepsilon_{i \cdot (ij)}^T \varepsilon_{j \cdot (ij)})}{(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{3/2}} \\ \frac{\varepsilon_{i \cdot (ij)}}{(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{1/2}} - \frac{\varepsilon_{j \cdot (ij)} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)}) (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})}{(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{3/2}} \end{bmatrix} \in \mathbb{R}^{2N}.$$

Moreover, the Hessian matrix of $f(\varepsilon_{ij})$ is equivalent to the Jacobian matrix of the gradient vector given by

$$H\{f(\varepsilon_{ij})\} = \begin{bmatrix} \frac{\partial^2 f}{\partial \varepsilon_{i \cdot (ij)} \partial \varepsilon_{i \cdot (ij)}} & \frac{\partial^2 f}{\partial \varepsilon_{j \cdot (ij)} \partial \varepsilon_{i \cdot (ij)}} \\ \frac{\partial^2 f}{\partial \varepsilon_{i \cdot (ij)} \partial \varepsilon_{j \cdot (ij)}} & \frac{\partial^2 f}{\partial \varepsilon_{j \cdot (ij)} \partial \varepsilon_{j \cdot (ij)}} \end{bmatrix} \in \mathbb{R}^{2N \times 2N}.$$

Letting $I_{N \times N}$ denote the $N \times N$ identity matrix, it follows that

$$\begin{aligned}
\frac{\partial^2 f}{\partial \varepsilon_{i \cdot (ij)} \partial \varepsilon_{i \cdot (ij)}} &= -2(\varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) \\
&\quad + 3(\varepsilon_{i \cdot (ij)} \varepsilon_{i \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-5/2} (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^2 (\varepsilon_{i \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) \\
&\quad - I_{N \times N} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)}), \\
\frac{\partial^2 f}{\partial \varepsilon_{j \cdot (ij)} \partial \varepsilon_{i \cdot (ij)}} &= \frac{\partial^2 f}{\partial \varepsilon_{i \cdot (ij)} \partial \varepsilon_{j \cdot (ij)}} = -(\varepsilon_{i \cdot (ij)} \varepsilon_{i \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) \\
&\quad + (\varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) \\
&\quad - (\varepsilon_{j \cdot (ij)} \varepsilon_{j \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)}) \\
&\quad + I_{N \times N} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-1/2}, \\
\frac{\partial^2 f}{\partial \varepsilon_{j \cdot (ij)} \partial \varepsilon_{j \cdot (ij)}} &= -2(\varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)}) \\
&\quad + 3(\varepsilon_{j \cdot (ij)} \varepsilon_{j \cdot (ij)}^T)(\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-5/2} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)})^2 (\varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) \\
&\quad - I_{N \times N} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{j \cdot (ij)}) (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)} \varepsilon_{j \cdot (ij)}^T \varepsilon_{j \cdot (ij)})^{-3/2} (\varepsilon_{i \cdot (ij)}^T \varepsilon_{i \cdot (ij)}).
\end{aligned}$$

3. CASE STUDY

We demonstrate the utility of our proposed inferential methods by analyzing data from the Autism Brain Imaging Data Exchange (ABIDE) initiative (Craddock et al., 2013). The ABIDE initiative has disseminated multi-site structural and functional brain imaging data to facilitate neuroimaging research of autism spectrum disorder. This neuroimaging research is important for improved understanding of brain activity patterns among individuals diagnosed with autism spectrum disorder to aid in earlier diagnosis and continued development of clinical treatments (Hull et al., 2017).

We focus our analysis on resting-state functional magnetic resonance imaging (fMRI) data from a single site provided by Stanford University that was preprocessed using the Data Processing Assistant for Resting-State fMRI pipeline. Within this site, we consider data from female participants diagnosed with autism spectrum disorder who were approximately 10 years old at the time of data acquisition. The Automated Anatomical Labeling atlas was used to extract data for 116 regions of interest, but we focus our analysis on ten brain regions in the Default Mode Network. See Craddock et al. (2013) for more details on data acquisition and preprocessing steps.

Since the resting-state fMRI data consists of 175 volumes for 10 regions of interest and the average autoregressive parameter estimated from a first-order autoregressive model for the data for each participant is about 0.40, the simulation setting that most aligns with our real data analysis has $N = 100$ observations, $p = 10$ variables, and $\phi = 0.4$ for the autoregressive parameter. With $p = 10$ variables, there are a total of 45 partial correlations for each participant. Partial correlation plots for all four participants showing the results of Wald tests of each individual partial correlation being equal to 0 or not are displayed in Figure 1, and Wald confidence intervals of each partial correlation are displayed in Table 1. After applying a multiplicity adjustment as proposed by Holm (1979), the partial correlation between the right and left cingulum posterior regions and the partial correlation between the right and left precuneus regions were statistically significant for all participants at the 5% significance level. These results align with previous studies exploring functional connectivity among brain regions in the default mode network (Hull et al., 2017).

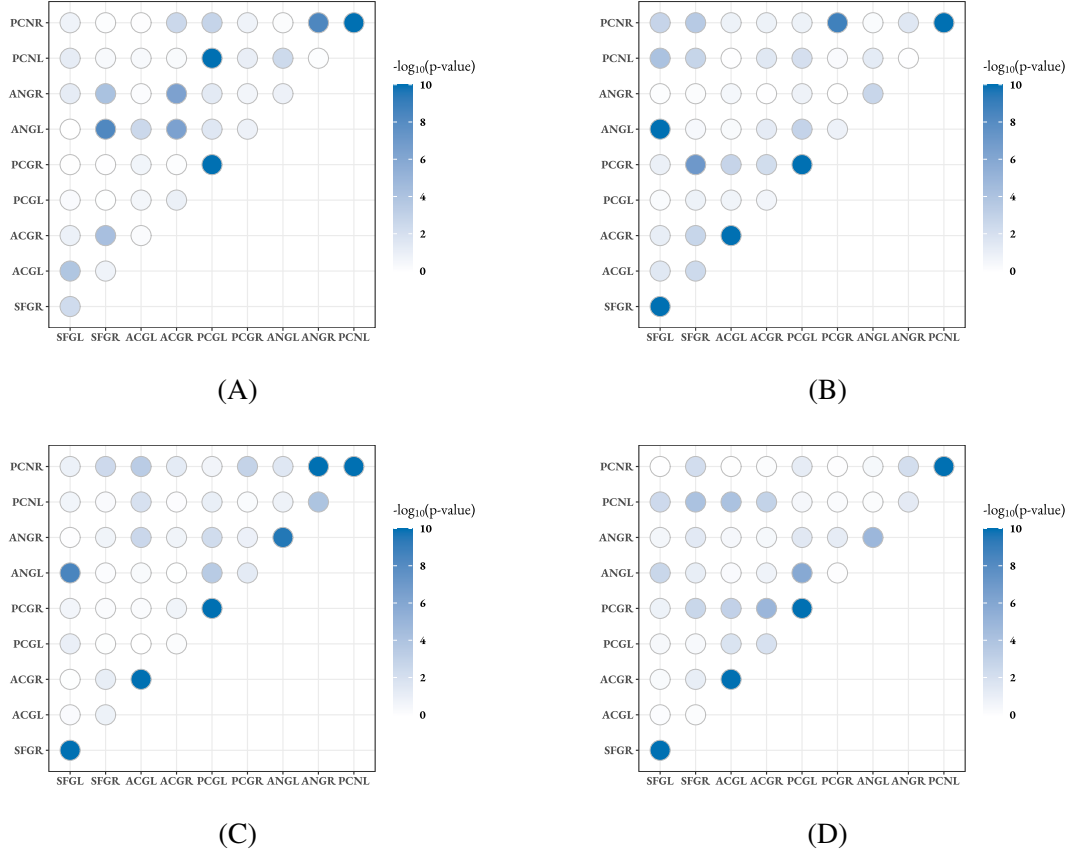


Fig. 1. Negative $\log_{10}$ p-values for Wald tests of null partial correlations. Each panel corresponds to the results for a different study participant.

Table 1. Wald 95% confidence intervals for individual partial correlations

Region 1	Region 2	Participant A	Participant B	Participant C	Participant D
SFGL	SFGR	(0 · 07, 0 · 38)	(0 · 56, 0 · 74)	(0 · 60, 0 · 77)	(0 · 49, 0 · 69)
SFGL	ACGL	(0 · 14, 0 · 43)	(0 · 02, 0 · 32)	(−0 · 10, 0 · 21)	(−0 · 11, 0 · 18)
SFGR	ACGL	(−0 · 27, 0 · 04)	(−0 · 36, −0 · 07)	(−0 · 29, 0 · 04)	(−0 · 11, 0 · 19)
SFGL	ACGR	(−0 · 03, 0 · 30)	(−0 · 32, 0 · 01)	(−0 · 14, 0 · 18)	(−0 · 09, 0 · 20)
SFGR	ACGR	(0 · 16, 0 · 45)	(0 · 09, 0 · 39)	(−0 · 01, 0 · 32)	(−0 · 01, 0 · 28)
ACGL	ACGR	(−0 · 11, 0 · 20)	(0 · 75, 0 · 86)	(0 · 84, 0 · 91)	(0 · 77, 0 · 87)
SFGL	PCGL	(−0 · 11, 0 · 22)	(−0 · 10, 0 · 20)	(−0 · 29, 0 · 02)	(−0 · 22, 0 · 08)
SFGR	PCGL	(−0 · 16, 0 · 17)	(−0 · 26, 0 · 03)	(−0 · 14, 0 · 18)	(−0 · 21, 0 · 09)
ACGL	PCGL	(−0 · 06, 0 · 25)	(−0 · 26, 0 · 05)	(−0 · 16, 0 · 15)	(0 · 03, 0 · 32)
ACGR	PCGL	(−0 · 03, 0 · 30)	(−0 · 06, 0 · 25)	(−0 · 12, 0 · 19)	(−0 · 33, −0 · 05)
SFGL	PCGR	(−0 · 13, 0 · 16)	(−0 · 29, 0 · 02)	(−0 · 06, 0 · 25)	(−0 · 28, 0 · 04)
SFGR	PCGR	(−0 · 17, 0 · 14)	(0 · 24, 0 · 51)	(−0 · 11, 0 · 21)	(0 · 09, 0 · 38)
ACGL	PCGR	(−0 · 05, 0 · 25)	(0 · 10, 0 · 40)	(−0 · 11, 0 · 20)	(−0 · 38, −0 · 10)
ACGR	PCGR	(−0 · 18, 0 · 13)	(−0 · 39, −0 · 07)	(−0 · 26, 0 · 05)	(0 · 16, 0 · 43)
PCGL	PCGR	(0 · 53, 0 · 72)	(0 · 36, 0 · 60)	(0 · 57, 0 · 74)	(0 · 58, 0 · 74)
SFGL	ANGL	(−0 · 16, 0 · 16)	(0 · 34, 0 · 59)	(0 · 29, 0 · 57)	(0 · 09, 0 · 40)
SFGR	ANGL	(0 · 27, 0 · 55)	(−0 · 22, 0 · 08)	(−0 · 22, 0 · 13)	(−0 · 30, 0 · 01)
ACGL	ANGL	(−0 · 37, −0 · 08)	(−0 · 09, 0 · 22)	(−0 · 22, 0 · 10)	(−0 · 21, 0 · 10)
ACGR	ANGL	(−0 · 52, −0 · 23)	(0 · 00, 0 · 31)	(−0 · 17, 0 · 15)	(−0 · 04, 0 · 25)
PCGL	ANGL	(0 · 02, 0 · 34)	(0 · 09, 0 · 38)	(0 · 13, 0 · 42)	(0 · 19, 0 · 45)
PCGR	ANGL	(−0 · 27, 0 · 03)	(−0 · 28, 0 · 03)	(−0 · 31, 0 · 00)	(−0 · 19, 0 · 14)
SFGL	ANGR	(0 · 00, 0 · 31)	(−0 · 14, 0 · 20)	(−0 · 14, 0 · 20)	(−0 · 27, 0 · 07)
SFGR	ANGR	(−0 · 43, −0 · 15)	(−0 · 12, 0 · 20)	(−0 · 30, 0 · 05)	(0 · 01, 0 · 33)
ACGL	ANGR	(−0 · 12, 0 · 19)	(−0 · 27, 0 · 08)	(0 · 09, 0 · 40)	(−0 · 22, 0 · 07)
ACGR	ANGR	(0 · 23, 0 · 51)	(−0 · 18, 0 · 20)	(−0 · 28, 0 · 05)	(−0 · 07, 0 · 22)
PCGL	ANGR	(−0 · 34, −0 · 01)	(−0 · 04, 0 · 28)	(−0 · 38, −0 · 07)	(−0 · 31, −0 · 02)
PCGR	ANGR	(−0 · 06, 0 · 26)	(−0 · 16, 0 · 18)	(−0 · 03, 0 · 28)	(0 · 00, 0 · 33)
ANGL	ANGR	(−0 · 03, 0 · 29)	(0 · 09, 0 · 39)	(0 · 30, 0 · 57)	(0 · 18, 0 · 48)
SFGL	PCNL	(0 · 00, 0 · 32)	(0 · 14, 0 · 42)	(−0 · 06, 0 · 27)	(0 · 08, 0 · 40)
SFGR	PCNL	(−0 · 09, 0 · 23)	(−0 · 37, −0 · 09)	(−0 · 24, 0 · 12)	(−0 · 44, −0 · 15)
ACGL	PCNL	(−0 · 23, 0 · 09)	(−0 · 16, 0 · 14)	(0 · 05, 0 · 36)	(0 · 14, 0 · 41)
ACGR	PCNL	(−0 · 10, 0 · 23)	(0 · 02, 0 · 32)	(−0 · 19, 0 · 13)	(−0 · 36, −0 · 09)
PCGL	PCNL	(0 · 39, 0 · 64)	(0 · 05, 0 · 34)	(−0 · 02, 0 · 29)	(−0 · 07, 0 · 22)
PCGR	PCNL	(−0 · 29, 0 · 01)	(−0 · 21, 0 · 10)	(−0 · 10, 0 · 21)	(−0 · 11, 0 · 21)
ANGL	PCNL	(0 · 08, 0 · 39)	(−0 · 31, 0 · 00)	(−0 · 04, 0 · 30)	(−0 · 13, 0 · 22)
ANGR	PCNL	(−0 · 14, 0 · 20)	(−0 · 17, 0 · 15)	(−0 · 48, −0 · 16)	(0 · 01, 0 · 34)
SFGL	PCNR	(−0 · 26, 0 · 05)	(−0 · 38, −0 · 09)	(−0 · 29, 0 · 03)	(−0 · 18, 0 · 16)
SFGR	PCNR	(−0 · 13, 0 · 19)	(0 · 12, 0 · 40)	(0 · 08, 0 · 40)	(0 · 06, 0 · 37)
ACGL	PCNR	(−0 · 14, 0 · 18)	(−0 · 27, 0 · 03)	(−0 · 43, −0 · 12)	(−0 · 14, 0 · 16)
ACGR	PCNR	(−0 · 40, −0 · 09)	(−0 · 04, 0 · 27)	(0 · 00, 0 · 32)	(−0 · 18, 0 · 11)
PCGL	PCNR	(−0 · 42, −0 · 10)	(−0 · 26, 0 · 03)	(−0 · 26, 0 · 05)	(0 · 00, 0 · 28)
PCGR	PCNR	(−0 · 04, 0 · 27)	(0 · 27, 0 · 53)	(0 · 10, 0 · 39)	(−0 · 19, 0 · 14)
ANGL	PCNR	(−0 · 21, 0 · 12)	(−0 · 10, 0 · 21)	(−0 · 34, −0 · 02)	(−0 · 25, 0 · 08)
ANGR	PCNR	(0 · 29, 0 · 57)	(0 · 02, 0 · 34)	(0 · 46, 0 · 70)	(0 · 07, 0 · 40)
PCNL	PCNR	(0 · 49, 0 · 71)	(0 · 58, 0 · 76)	(0 · 66, 0 · 81)	(0 · 33, 0 · 60)

SFGL, Frontal Superior Medial Left; SFGR, Frontal Superior Medial Right; ACGL, Cingulum Anterior Left; ACGR, Cingulum Anterior Right; PCGL, Cingulum Posterior Left; PCGR, Cingulum Posterior Right; ANGL, Angular Left; ANGR, Angular Right; PCNL, Precuneus Left; PCNR, Precuneus Right

REFERENCES

Craddock, C., Benhajali, Y., Chu, C., Chouinard, F., Evans, A., Jakab, A., Khundrakpam, B. S., Lewis, J. D., Li, Q., Milham, M., Yan, C., and Bellec, P. (2013). The neuro bureau preprocessing initiative: Open sharing of

preprocessed neuroimaging data and derivatives. *Neuroinformatics*.

Drygas, H. (1976). Weak and strong consistency of the least squares estimators in regression models. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 34:119–127.

120 Holm, S. (1979). A simple sequentially rejective multiple test procedure. *Scandinavian Journal of Statistics*, 6:65–70.

Hull, J. V., Dokovna, L. B., Jacokes, Z. J., Torgerson, C. M., Irimia, A., and Van Horn, J. D. (2017). Resting-state functional connectivity in autism spectrum disorders: a review. *Frontiers in Psychiatry*, 7.

McMurry, T. L. and Politis, D. N. (2010). Banded and tapered estimates for autocovariance matrices and the linear process bootstrap. *Journal of Time Series Analysis*, 31(6):471–482.

125 Roy, R. (1989). Asymptotic covariance structure of serial correlations in multivariate time series. *Biometrika*, 76(4):824–827.

Roy, R. and Cléroux, R. (1993). Vector cross-correlation in time series and applications. *International Statistical Review*, 61(3):447–464.

Samorodnitsky, G. (2016). *Stochastic processes and long range dependence*. Springer series in operations research. Springer, Cham.

130 Wu, W. B. and Pourahmadi, M. (2009). Banding sample autocovariance matrices of stationary processes. *Statistica Sinica*, 19(4):1755–1768.

[Received 1 August 2022. Editorial decision on 1 September 2022]