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Inference of partial correlations of a multivariate Gaussian time series

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SUMMARY

We derive an asymptotic joint distribution and novel covariance estimator for the partial correlations of a multivariate Gaussian time series given mild regularity conditions. Using our derived asymptotic distribution, we develop a Wald confidence interval and testing procedure for inference of individual partial correlations for time series data. Through simulation we demonstrate that our proposed confidence interval attains higher coverage rates, and our testing procedure attains false positive rates closer to the nominal levels than approaches that assume independent observations when autocorrelation is present.

Some key words: Autocorrelation; Partial correlation; Quadratic form; Taylor series.

1. INTRODUCTION

The partial correlation coefficient measures the strength of the linear relationship between two variables of interest after removing the effect of other variables. Under an assumption of independent observations, the partial correlation has been commonly used to describe conditional dependencies in areas such as geoscience (Erb, 2020) and genomics (de la Fuente et al., 2004). For independent and normally distributed data, the sampling distribution of a single partial correlation coefficient and the asymptotic joint distribution of partial correlations have been derived (Fisher, 1924; Hedges and Olkin, 1983). Based on these derived distributions many inferential methods for partial correlations have been proposed (Cramer, 1974). The assumption of independent observations that these methods rely on is violated in time series data where observations are correlated over time, possibly resulting in spurious correlations (Student, 1914).

For time series data, the partial correlation coefficient has been used in areas such as ecology (Damos, 2016), economics (Kenett et al., 2015), and neuroimaging (Marrelec et al., 2006). Several asymptotic results have been provided for the partial autocorrelation, the correlation of a time series with its own lagged values after removing the linear effect of shorter lags. Under a univariate autoregressive model, the asymptotic distribution and standard error of the partial autocorrelation has been derived (Barndorff-Nielsen and Schou, 1973). For a weakly stationary univariate time series, the joint distribution and second-order properties of the partial autocorrelations have also been provided (Stoica, 1989). For multivariate time series data, the elements of the lag m partial autocorrelation matrix for $m > p$ under a pth-order autoregressive model have been shown to be asymptotically normal with mean 0 (Ansley and Newbold, 1979). In a non-null setting, the joint distribution for the empirical marginal correlations of a general multivariate time series has been provided (Roy, 1989), however, partial correlations can yield different results than marginal correlations since they account for potentially confounding variables in settings such as graphical modeling in neuroimaging (Kim et al., 2015). Qiu and Zhou (2022) provided the asymptotic distribution for an individual partial correlation based on a first-order series expansion for time series data, but they do not provide the joint distribution or the asymptotic covariance between pairs of empirical partial correlations. The joint distribution of the empirical partial correlations for time series data is important for hierarchical models in high-dimensional settings such as neuroimaging, but has not been provided to the best of our knowledge.

Many methods of inference for the contemporaneous partial correlations of time series data have also been proposed. Some have proposed pre-whitening data to remove autocorrelation prior to conducting inference (Haugh, 1976) or using modified standard error estimators (Cliff et al., 2021). In neuroimaging, several tests of partial correlations for Gaussian graphical models have also been proposed for multivariate time series data (Qiu et al., 2016; Qiu and Zhou, 2022). A common limitation of these approaches is the reliance on correctly specifying a model for autocorrelation present in the data (Box and Newbold, 1971). Moreover, these approaches provide inferential methods for partial correlations under an assumed null setting of the true partial correlations being 0, but do not provide confidence intervals or the joint distribution of the empirical partial correlations for a multivariate time series in a non-null setting.

Knowledge of the asymptotic joint distribution for the partial correlations of a multivariate Gaussian time series would facilitate construction of confidence intervals in a non-null setting. Confidence intervals are important in areas such as graphical modeling in neuroimaging for quantifying uncertainty in the strength of partial correlations. The joint distribution of the partial correlations would also facilitate use of multi-level models which are useful for overcoming low signal-to-noise ratios in areas such as neuroimaging and other high-dimensional settings. To this end, we derive an asymptotic distribution for the partial correlations of a weakly stationary multivariate Gaussian time series. We also provide an explicit form for the asymptotic covariance structure of the partial correlations and propose a consistent estimator for this covariance. This is completed using a second-order Taylor series approximation and properties of quadratic forms of multivariate random vectors. Based on this derived distribution, we propose a Wald confidence interval and testing procedure for inference of the contemporaneous partial correlations that does not rely on correctly specifying a model for the autocorrelation structure of the data. Our proposed methods of inference pertain to the contemporaneous relationships of a multivariate time series, and these relationships are not conditioned on observations at previous time points. However, our proposed inferential methods still account for the autocorrelation present in the multivariate time series. We show through simulations the advantage of our proposed inferential procedures compared to others that assume independent observations, achieving closer to nominal coverage and false positive rates for multivariate time series data where autocorrelation is present without imposing strong assumptions on the autocorrelation structure. Proofs of all theorems are provided in the Supplementary Material.

2. Asymptotics of Sample Partial Correlations

2·1. *Asymptotic Distribution*

Let $x(t) = \{x_k(t)\}_{k=1}^p$ be a p-variate time series that is second-order stationary and ergodic, and $\{x_k\}_{k=1}^p$ be an N-length realization of $x(t)$ such that $x_k \in \mathbb{R}^N$ for $k = 1, \ldots, p$. Generally, a multivariate time series is second-order stationary if it has a constant mean, and its covariance function $\gamma\{x_i(t-\ell), x_j(t)\} = \gamma_{ij}(\ell)$ depends only on the lag, ℓ, between points. Letting $e_{i\cdot(ij)}$ and $e_{j\cdot(ij)}$ be the N-length vectors of ordinary least squares residuals from contemporaneously regressing x_i and x_j respectively on the other $p-2$ variables $\{x_k\}_{k\neq i,j}$, it follows that one way to express the empirical partial correlation between x_i and x_j is

$$r_{ij\cdot(ij)} = f(e_{ij}) = e_{i\cdot(ij)}^T e_{j\cdot(ij)} (e_{i\cdot(ij)}^T e_{i\cdot(ij)} e_{j\cdot(ij)}^T e_{j\cdot(ij)})^{-1/2}, \tag{1}$$

where $e_{ij} = \left[e_{i\cdot(ij)}^T, e_{j\cdot(ij)}^T\right]^T$, and $r_{ij\cdot(ij)}$ is equivalent to the sample marginal correlation between $e_{i\cdot(ij)}$ and $e_{j\cdot(ij)}$. To derive the asymptotic joint distribution of the empirical partial correlations $\{r_{ij\cdot(ij)}\}_{i\neq j}$ of $\{x_k\}_{k=1}^p$ in Theorem 1, we consider the following conditions:

1. The spectral density functions of each component of $x(t)$ are square-integrable.
2. The following conditions hold for each component of $x(t)$:
   a. $x(t) = \sum_{\ell=0}^{\infty} A(\ell)\varepsilon(t-\ell)$, where $\varepsilon(t-\ell)$ is the vector of one-step linear prediction residuals at lag ℓ and $A(\ell) \in \mathbb{R}^{p\times p}$;
   b. $E\{\varepsilon(t_1)\varepsilon(t_2)^T\} = \begin{cases} 0, & t_1 \neq t_2 \\ \Sigma, & t_1 = t_2 \end{cases}$, where $\Sigma \in \mathbb{R}^{p\times p}$ is nonsingular;
   c. $E\{\varepsilon(t)\} = 0$;
   d. $\sum_{\ell=0}^{\infty} ||A(\ell)||^2 < \infty$, where $||\cdot||$ is the Euclidean norm.
3. The first through fourth moments of $\varepsilon(t|\mathcal{F}_{t-1})$ are all finite constants, where $\mathcal{F}_{t-1}$ is the sub σ-algebra generated by $\{x(t') : t' < t\}$.

Conditions 1 through 3 are necessary for a result regarding the asymptotic normality of the marginal correlations and serial covariances of a multivariate time series (Roy, 1989; Hannan, 1976). With these conditions, we present our main result in Theorem 1.

THEOREM 1. *Let $x(t) = \{x_k(t)\}_{k=1}^p$ be an ergodic, second-order stationary p-variate time series satisfying Conditions 1 through 3 above, and $\{x_k\}_{k=1}^p$ be an N-length realization of $x(t)$. Then if $r_{ij\cdot(ij)}$ is the empirical partial correlation between x_i and x_j, it follows that $N^{1/2}(r_{ij\cdot(ij)} - \rho_{ij\cdot(ij)})$ converges in distribution to a normal with mean 0 where $\rho_{ij\cdot(ij)}$ is the population partial correlation for all $i \neq j$.*

A proof of Theorem 1 is provided in Section 1 of the Supplementary Material.

2·2. *Asymptotic Covariance Estimator*

To derive the asymptotic variance of a single partial correlation and the asymptotic covariances of the empirical partial correlations in Theorem 2, we utilize the representation of the partial correlation between any pair of variables as a function of the ordinary least squares residuals. We approximate this function, $f(e_{ij})$ in Equation (1), using a second-order Taylor series expansion around $\varepsilon_{ij} = \left[\varepsilon_{i\cdot(ij)}^T, \varepsilon_{j\cdot(ij)}^T\right]^T$, the theoretical residuals such that $\rho_{ij\cdot(ij)} = f(\varepsilon_{ij})$, as

$$f(e_{ij}) \approx f(\varepsilon_{ij}) + (e_{ij} - \varepsilon_{ij})^\mathsf{T} \nabla f(\varepsilon_{ij}) + 1/2(e_{ij} - \varepsilon_{ij})^\mathsf{T} H\{f(\varepsilon_{ij})\}(e_{ij} - \varepsilon_{ij}),$$

where $\nabla f(\varepsilon_{ij}) = E\{\nabla f(e_{ij})\} \in \mathbb{R}^{2N}$ is the expected value of the gradient of $f(e_{ij})$, and $H\{f(\varepsilon_{ij})\} = E\{H(e_{ij})\} \in \mathbb{R}^{2N\times 2N}$ is the expected value of the Hessian matrix of $f(e_{ij})$. Explicit forms of $\nabla f(\varepsilon_{ij})$ and $H\{f(\varepsilon_{ij})\}$ are derived in Section 2 of the Supplementary Material. For Theorem 2, we also assume that $x(t)$ is multivariate Gaussian and that for all $i \neq j$:

4. $E(\varepsilon_{i\cdot(ij)}) = 0$ and $E(\varepsilon_{i\cdot(ij)}\varepsilon_{i\cdot(ij)}^T) = \Sigma_{i\cdot(ij)} \in \mathbb{R}^{N\times N}$;

5. $\lambda_{\max}(\Sigma_{i\cdot(ij)}) < \infty$, where $\lambda_{\max}(\Sigma_{i\cdot(ij)})$ is the largest eigenvalue of $\Sigma_{i\cdot(ij)}$;
6. $\lambda^{+}_{\min}(X^T_{(ij)}X_{(ij)})$ converges to ∞ almost surely, where $\lambda^{+}_{\min}(X^T_{(ij)}X_{(ij)})$ is the smallest positive eigenvalue of $X^T_{(ij)}X_{(ij)}$ and $X_{(ij)} \in \mathbb{R}^{N\times(p-2)}$ is the design matrix excluding x_i and x_j.

Conditions 4 through 6 ensure consistency of the least squares estimator for stationary linear processes (Drygas, 1976). The asymptotic covariances of the partial correlations can be expressed using a more complex form of the fourth-order moments of the empirical residuals without the Gaussian assumption. However, this simplifies the covariance structure by using properties of quadratic forms of Gaussian random vectors. In this setting, we present Theorem 2 which provides the asymptotic variance and covariances for the empirical partial correlations.

THEOREM 2. *Assume $x(t)$ satisfies Conditions 1 through 6 and that $x(t)$ is multivariate Gaussian. Then the asymptotic variance of $r_{ij\cdot(ij)}$ is $\tilde{\gamma}_{ij} = 1/2\,(tr\,[H\{f(\varepsilon_{ij})\}\Sigma_{ij}H\{f(\varepsilon_{ij})\}\Sigma_{ij}])$, where $tr(\cdot)$ denotes the trace function and $\Sigma_{ij} = cov(e_{ij})$. Moreover, the asymptotic covariance between $r_{ij\cdot(ij)}$ and $r_{km\cdot(km)}$ is $\tilde{\gamma}_{ijkm} = 1/2tr\left[H\{f(\varepsilon_{ij})\}\Sigma_{ijkm12}H\{f(\varepsilon_{km})\}\Sigma^T_{ijkm12}\right]$, where $\Sigma_{ijkm12} = cov(e_{ij}, e_{km})$.*

A proof of Theorem 2 is provided in Section 1 of the Supplementary Material. In the proof, we approximate $r_{ij\cdot(ij)}$ and $r_{km\cdot(km)}$ as quadratic forms of Gaussian random vectors which are asymptotically distributed as a generalized χ^2 distribution (Imhof, 1961). However, to facilitate more general inference we apply the conditions and result of Theorem 1 to obtain that the empirical partial correlations are asymptotically normal.

We show in the Supplementary Material that $\tilde{\gamma}_{ij}$ can be expressed as the trace of a fourth-order matrix polynomial of Σ_{ii}, Σ_{jj}, and Σ_{ij}. As a higher amount of autocorrelation is present in $e_{i\cdot(ij)}$ and $e_{j\cdot(ij)}$ as described by the autocovariance matrices Σ_{ii} and Σ_{jj}, respectively, the size of $\tilde{\gamma}_{ij}$ will increase. Since $\tilde{\gamma}_{ij}$ is the asymptotic variance of $r_{ij\cdot(ij)}$, $\tilde{\gamma}_{ij}^{-1/2}||r_{ij\cdot(ij)} - \rho_{ij\cdot(ij)}|| = O_p(1)$ (Serfling, 1980). Thus, the more autocorrelation present in $e_{i\cdot(ij)}$ and $e_{j\cdot(ij)}$, the slower the convergence rate of $r_{ij\cdot(ij)}$. Since the covariance $\tilde{\gamma}_{ijkm}$ is a generalization of the variance $\tilde{\gamma}_{ij}$, we focus the discussion of our empirical estimation on $\tilde{\gamma}_{ijkm}$. To estimate the matrices constituting $\tilde{\gamma}_{ijkm}$, we use tapered covariance estimators for the blocks of the covariance matrices (McMurry and Politis, 2010) and method of moments estimators for the Hessian matrices based on the empirical ordinary least squares residuals to form our proposed estimator, $\hat{\gamma}_{ijkm} = 1/2\text{tr}\left[H\{f(e_{ij})\}\hat{\Sigma}_{ijkm12}H\{f(e_{km})\}\hat{\Sigma}^T_{ijkm12}\right]$. We establish the consistency of our estimator in Theorem 3 by using Conditions 1 through 3 to ensure the tapered covariance matrices are consistent estimators of the covariance matrices and Conditions 4 through 6 to ensure the method of moments estimators are consistent estimators of the Hessian matrices.

THEOREM 3. *Let $x(t) = \{x_k(t)\}_{k=1}^p$ be an ergodic, second-order stationary p-variate time series satisfying Conditions 1 through 6 above. Then $\hat{\gamma}_{ijkm}$ is a consistent estimator for $\tilde{\gamma}_{ijkm}$.*

A proof of Theorem 3 is provided in Section 1 of the Supplementary Material.

3. Inferential Methods

We demonstrate the utility of our derived distribution and asymptotic covariance structure in finite samples by implementing a Wald confidence interval for individual partial correlations. Specifically, we calculate the $100 \times (1-\alpha)\%$ Wald confidence intervals for each $\rho_{ij\cdot(ij)}$, the population-level partial correlation, as $r_{ij\cdot(ij)} \pm Z_{\alpha/2} \times SE(r_{ij\cdot(ij)})$, where $Z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution and $SE(r_{ij\cdot(ij)})$ is the standard error of $r_{ij\cdot(ij)}$ for i, $j = 1, \ldots, p$. The approximate standard error based on our second-order Taylor series approximation for $r_{ij\cdot(ij)}$ is $\tilde{\gamma}_{ij}^{1/2}$ which we estimate with $\hat{\gamma}_{ij}^{1/2}$.

To test whether individual partial correlations are 0 or not, i.e., H_0: $\rho_{ij\cdot(ij)} = 0$ vs. H_A: $\rho_{ij\cdot(ij)} \neq 0$, we propose a Wald testing procedure. The corresponding Wald test statistic is $W_{ij} = r^2_{ij\cdot(ij)}/\hat{\gamma}_{ij}$. This procedure is equivalent to using a likelihood ratio test, and thus under the null hypothesis W_{ij} asymptotically follows a χ^2 distribution with 1 degree of freedom (Wilks, 1938). We reject the null hypothesis if $W_{ij} \geq \chi^2_{(1;1-\alpha)}$, where $\chi^2_{(1;1-\alpha)} \approx 3.84$ for $\alpha = 0.05$.

4. Simulations

4·1. *Setup*

We implemented simulations to assess the performance of the confidence intervals and inferential procedures using our derived asymptotic distribution. A total of 1,000 data sets were generated for each setting with N observations of a p-variate first-order autoregressive model with correlation parameter ϕ. We considered three different numbers of variables ($p \in \{5, 10, 15\}$), two different sample sizes ($N \in \{100, 500\}$), three different levels of autocorrelation ($\phi \in \{0, 0\cdot 40, 0\cdot 80\}$), and generated partial correlations either uniformly from the set $\{-0\cdot 30, 0, 0\cdot 30\}$ or as all being 0, yielding 36 unique settings in total. Simulations were conducted using R (R Core Team, 2021).

We considered three other competing approaches to compare the performance of the confidence intervals: a naïve confidence interval assuming normally distributed and independent observations, a Fisher-transformed interval also assuming normally distributed and independent observations, and a block-bootstrap interval for each partial correlation. For the naïve approach, the estimated standard error for $r_{ij\cdot(ij)}$ is $(1 - r^2_{ij\cdot(ij)})^{1/2}(N-p)^{-1/2}$ (Cramer, 1974). Thus, a naïve 95% confidence interval for $\rho_{ij\cdot(ij)}$ is $r_{ij\cdot(ij)} \pm t^*_{(N-p)}(1 - r^2_{ij\cdot(ij)})^{1/2}(N-p)^{-1/2}$ where $t^*_{(N-p);\alpha/2}$ is the $\alpha/2$ quantile of a t-distribution with $N-p$ degrees of freedom. We constructed Fisher-transformed confidence intervals centered around the inverse hyperbolic tangent of the partial correlations as $\tanh^{-1}(r_{ij\cdot(ij)}) = 1/2 \log\{(1 + r_{ij\cdot(ij)})/(1 - r_{ij\cdot(ij)})\}$, and under an assumption of independent and normally distributed observations $\tanh^{-1}(r_{ij\cdot(ij)})$ converges to a normal distribution with mean $\tanh^{-1}(\rho_{ij\cdot(ij)})$ and variance $1/(N-p-1)$ (Cramer, 1974; Fisher, 1915). Thus, we constructed confidence intervals for $\tanh^{-1}(\rho_{ij\cdot(ij)})$ centered around $\tanh^{-1}(r_{ij\cdot(ij)})$ using an estimated standard error of $(N-p-1)^{-1/2}$, and then transformed the endpoints using the hyperbolic tangent function to obtain a corresponding interval for $\rho_{ij\cdot(ij)}$. For the block-bootstrap, we used the $\alpha/2$ and $1-(\alpha/2)$ quantiles calculated from 1,000 bootstrap samples and selected the block-length using an automatic selection algorithm for stationary multivariate time series data (Politis and White, 2004).

We also compared the performance of our proposed Wald test to a naïve t-test, a hypothesis test based on a Fisher transformation, and a block-bootstrap testing procedure. The naïve t-test statistic is $t_{ij} = r_{ij\cdot(ij)}(N-p)^{1/2}(1 - r^2_{ij\cdot(ij)})^{-1/2}$ which we compared to the quantiles of a t-distribution with $N-p$ degrees of freedom (Levy and Narula, 1978). The Fisher-transformed test statistic is $Z_{ij} = \tanh^{-1}(r_{ij\cdot(ij)})(N-p-1)^{-1/2}$ which we compared to the quantiles of a standard normal distribution. For the block-bootstrap, we rejected the null hypothesis if 0 was outside the $\alpha/2$ and $1-\alpha/2$ quantiles of the 1,000 generated bootstrap samples.

4·2. *Results*

Simulation results are summarized in Tables 1 and 2. Table 1 displays the coverage rates of 95% confidence intervals for individual partial correlations, and the largest standard error across all methods and settings was $0 \cdot 01$. All four approaches considered achieved close to the nominal coverage rate of 95% for the independence setting ($\phi = 0$). As expected, when autocorrelation was present the block-bootstrap and our proposed Wald interval attained coverage rates closer to the nominal rate than the naïve and Fisher transformation intervals which assumed independent observations. As the sample size increased, the Wald and block-bootstrap intervals approached the nominal coverage rate of 95% for all settings. The Fisher transformation and naïve approaches, however, attained lower coverage rates as the sample size increased when a moderate to high amount of autocorrelation was present ($\phi \in \{0 \cdot 4, 0 \cdot 8\}$).

Table 2 displays the results of testing for a zero partial correlation for each of the p choose 2 unique partial correlations comparing our approach to the competing methods. The largest standard error across all methods, metrics, and settings was $0 \cdot 01$. From Table 2 we observe that our Wald test and the block-bootstrap performed similarly in terms of Matthews correlation coefficient, and both outperformed the Fisher transformation and naïve approaches when autocorrelation was present, but still achieved similar Matthews correlation coefficient values in the independence setting. Our Wald testing procedure and the block-bootstrap also outperformed the Fisher and naïve testing approaches in terms of the false positive rate when autocorrelation was present, but still attained similar false positive rates in the independence case. The naïve and Fisher transformation approaches achieved higher true positive rates in the high-autocorrelation settings ($\phi = 0 \cdot 8$), but yielded inflated false positive rates in these settings compared to our Wald testing procedure. The only setting in which there was a meaningful difference between the block-bootstrap and our Wald test was the high autocorrelation ($\phi = 0 \cdot 8$) low sample size ($N = 100$) case, in which the block-bootstrap achieved somewhat lower false positive rates. This could be due to the convergence rate of the partial correlation being slower when a higher amount of autocorrelation is present as mentioned at the end of Section 2·2. For the large sample size setting ($N = 500$) the methods performed similarly in terms of false positive rates. When implemented properly, the block-bootstrap provides a flexible approach for inference across a variety of settings. However, it can be computationally expensive in high-dimensional settings. Advantages of our approach compared to the block-bootstrap are potential gains in computational efficiency and our derived asymptotic distribution can more easily be integrated into other modeling frameworks such as hierarchical models for multi-subject analysis.

Table 1. *Coverage rates based on 1,000 simulations for* 95% *confidence intervals of individual partial correlations*

		$N = 100$				$N = 500$			
p	ϕ	Wald	Bootstrap	Fisher	Naïve	Wald	Bootstrap	Fisher	Naïve
5	0	$0 \cdot 94$	$0 \cdot 93$	$0 \cdot 94$	$0 \cdot 96$	$0 \cdot 95$	$0 \cdot 95$	$0 \cdot 95$	$0 \cdot 96$
	$0 \cdot 4$	$0 \cdot 92$	$0 \cdot 92$	$0 \cdot 90$	$0 \cdot 92$	$0 \cdot 94$	$0 \cdot 93$	$0 \cdot 90$	$0 \cdot 92$
	$0 \cdot 8$	$0 \cdot 83$	$0 \cdot 87$	$0 \cdot 65$	$0 \cdot 69$	$0 \cdot 92$	$0 \cdot 91$	$0 \cdot 64$	$0 \cdot 66$
10	0	$0 \cdot 94$	$0 \cdot 94$	$0 \cdot 94$	$0 \cdot 95$	$0 \cdot 91$	$0 \cdot 91$	$0 \cdot 91$	$0 \cdot 93$
	$0 \cdot 4$	$0 \cdot 92$	$0 \cdot 92$	$0 \cdot 90$	$0 \cdot 92$	$0 \cdot 92$	$0 \cdot 91$	$0 \cdot 87$	$0 \cdot 89$
	$0 \cdot 8$	$0 \cdot 80$	$0 \cdot 89$	$0 \cdot 67$	$0 \cdot 71$	$0 \cdot 91$	$0 \cdot 90$	$0 \cdot 63$	$0 \cdot 66$
15	0	$0 \cdot 91$	$0 \cdot 91$	$0 \cdot 91$	$0 \cdot 92$	$0 \cdot 73$	$0 \cdot 71$	$0 \cdot 73$	$0 \cdot 75$
	$0 \cdot 4$	$0 \cdot 89$	$0 \cdot 90$	$0 \cdot 87$	$0 \cdot 88$	$0 \cdot 77$	$0 \cdot 75$	$0 \cdot 70$	$0 \cdot 72$
	$0 \cdot 8$	$0 \cdot 77$	$0 \cdot 90$	$0 \cdot 69$	$0 \cdot 72$	$0 \cdot 84$	$0 \cdot 84$	$0 \cdot 56$	$0 \cdot 58$

N, number of observations; p, number of variables; ϕ, autocorrelation parameter.

Table 2. *Results based on 1,000 simulations for testing if each individual partial correlation is nonzero*

			$N = 100$				$N = 500$			
Metric	p	ϕ	Wald	Bootstrap	Fisher	Naïve	Wald	Bootstrap	Fisher	Naïve
TPR	5	0	0·87	0·86	0·86	0·87	1·00	1·00	1·00	1·00
		0·4	0·80	0·79	0·82	0·83	1·00	1·00	1·00	1·00
		0·8	0·58	0·53	0·72	0·73	0·93	0·94	0·99	0·99
	10	0	0·77	0·75	0·75	0·77	1·00	1·00	1·00	1·00
		0·4	0·70	0·68	0·72	0·74	1·00	1·00	1·00	1·00
		0·8	0·55	0·46	0·65	0·66	0·89	0·89	0·98	0·98
	15	0	0·58	0·53	0·55	0·57	1·00	1·00	1·00	1·00
		0·4	0·54	0·49	0·54	0·57	0·99	0·99	0·99	0·99
		0·8	0·48	0·34	0·54	0·56	0·77	0·77	0·93	0·93
FPR	5	0	0·06	0·06	0·05	0·06	0·05	0·05	0·05	0·05
		0·4	0·07	0·08	0·09	0·10	0·06	0·06	0·10	0·10
		0·8	0·17	0·12	0·34	0·35	0·08	0·10	0·36	0·36
	10	0	0·06	0·06	0·05	0·06	0·05	0·05	0·05	0·05
		0·4	0·08	0·08	0·09	0·10	0·06	0·06	0·09	0·10
		0·8	0·20	0·11	0·31	0·33	0·09	0·09	0·35	0·35
	15	0	0·06	0·05	0·05	0·06	0·05	0·06	0·05	0·05
		0·4	0·08	0·07	0·09	0·10	0·06	0·07	0·09	0·10
		0·8	0·21	0·09	0·28	0·29	0·10	0·09	0·35	0·35
MCC	5	0	0·82	0·81	0·82	0·82	0·95	0·95	0·96	0·95
		0·4	0·74	0·73	0·75	0·75	0·95	0·94	0·92	0·91
		0·8	0·44	0·45	0·40	0·39	0·86	0·85	0·68	0·68
	10	0	0·69	0·67	0·68	0·69	0·96	0·96	0·96	0·96
		0·4	0·61	0·60	0·62	0·62	0·95	0·95	0·93	0·92
		0·8	0·34	0·37	0·33	0·33	0·79	0·80	0·69	0·69
	15	0	0·54	0·51	0·52	0·54	0·95	0·95	0·96	0·95
		0·4	0·48	0·45	0·48	0·48	0·94	0·93	0·91	0·91
		0·8	0·28	0·30	0·27	0·27	0·66	0·67	0·61	0·61

TPR, True positive rate; FPR, false positive rate; MCC, Matthews correlation coefficient.

Supplementary material

Supplementary material available at Biometrika online contains proofs of Theorems 1 through 3 and a case study to demonstrate the utility of our proposed methods. An R package implementing our derived asymptotic covariance estimator is available at `https://github.com/dilernia/pcCov`.

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